EXERCISES [MAI 5.8]

MONOTONY – CONCAVITY – OPTIMIZATION

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (i) $f'(x)=5x^4+e^x>0$, f increasing (ii) $f'(x)=3x^2+\frac{1}{x}>0$, f increasing

(iii) $f'(x)=-6e^{2x}<0$, f decreasing (iv) $f'(x)=\frac{2e^x}{(e^x+1)^2}>0$, f increasing

2. (a) $f'(x)=x^2+4x-5$

(b) $f'(x)=0 \Leftrightarrow x=-5, x=1$

so $x=-5$

(c) $f''(x)=2x+4$

$2x+4=0$

$x=-2$

(d) $x=-5$

3. $x=1$ and $x=3$ are points of inflection ($x=4$ is not)

4. (a) $f''(x)=0$ **OR** the max and min of f' gives the points of inflexion on f

–0.114, 0.364

(b) graph of g is a quadratic function, so it does not have any points of inflexion

OR graph of g is concave down over entire domain therefore no change in concavity

OR $g''(x)=-144$, therefore no points of inflexion as $g''(x)\neq 0$

5. (a)

Interval	g'	g''
$a<x<b$	**positive**	**positive**
$e<x<f$	**negative**	**negative**

(b)

Conditions	Point
$g'(x)=0$, $g''(x)<0$	**C**
$g'(x)<0$, $g''(x)=0$	**D**

(c) $g'(e)=0$ $g''(e)=0$

(d) 3 points of inflection (one of them, E, is a stationary point of inflection)

6. (a) (i) $x = -\frac{5}{2}$ (ii) $y = \frac{3}{2}$

(b) By quotient rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x+5)(3)-(3x-2)(2)}{(2x+5)^2} = \frac{19}{(2x+5)^2}$

(c) There are no stationary points, since $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ (or by the graph) (A1)

(d) There are no points of inflexion.

7. (a) $f''(x) = 3(x-3)^2$

(b) $f'(3) = 0, f''(3) = 0$

(c) f'' does not change sign at P

8. (a) $f'(x) = 2x\mathrm{e}^{-x} - x^2\mathrm{e}^{-x} = (2-x)\,x\,\mathrm{e}^{-x}$

(b) Maximum occurs at $x = 2$

Exact maximum value = $4\mathrm{e}^{-2}$

(c) $f''(x) = 2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} + x^2\mathrm{e}^{-x} = (x^2 - 4x + 2)e^{-x}$

For inflexion, $f''(x) = 0$

$x = \frac{4+\sqrt{8}}{2}\ \left(= 2+\sqrt{2}\right)$

9. (a) $g'(x) = \frac{x^2\left(\frac{1}{x}\right) - 2x\ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$

(b) $g'(x) = 0 \Leftrightarrow 1 - 2\ln x = 0 \Leftrightarrow \ln x = \frac{1}{2} \Leftrightarrow x = \mathrm{e}^{\frac{1}{2}}$

10. (a) x-intercepts at –3, 0, 2

(b) $-3 < x < 0,\ 2 < x < 3$

(c) the graph of f is **concave-down** therefore the second derivative is negative

11. (a) $x = 1$

(b) Using quotient rule

$h'(x) = \frac{(x-1)^2(1) - (x-2)[2(x-1)]}{(x-1)^4} = \frac{(x-1)-(2x-4)}{(x-1)^3} = \frac{3-x}{(x-1)^3}$

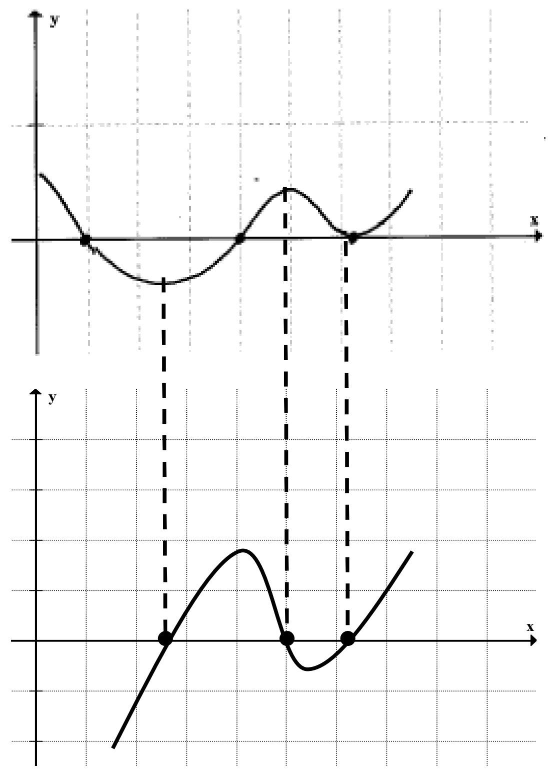
(c) at point of inflexion $g''(x) = 0$

$x = 4$

$y = \frac{2}{9} = 0.222$ *ie* P$\left(4, \frac{2}{9}\right)$

GRAPH OF f'

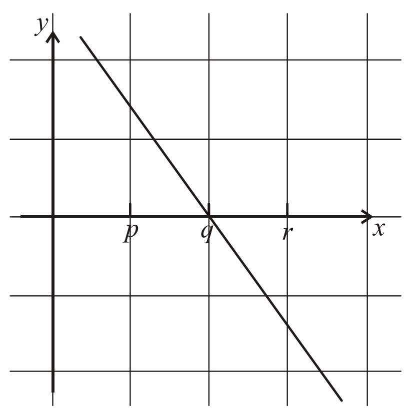
12.



13.

Graph	$f'(x)$	$f''(x)$
Diagram	I	IV

14. (a)



Note: *negative gradient, x- intercept of q. (It need not be linear)*

(b)

	Maximum point on f	Inflexion point on f
x-coordinate	r	q

(c) **METHOD 1**
Second derivative is zero, second derivative changes sign.
METHOD 2
There is a maximum on the graph of the first derivative.

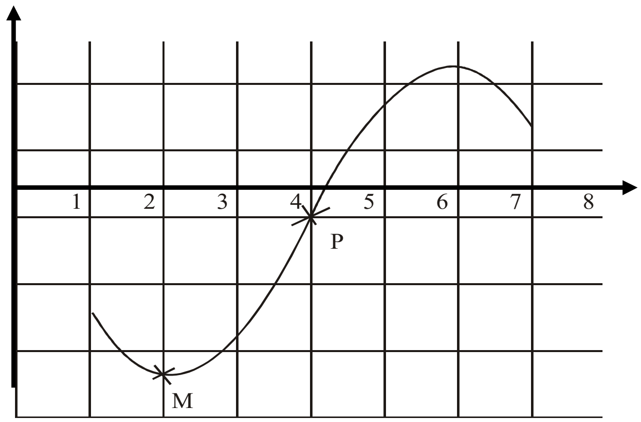
15. (a) $x = 4$ g'' changes sign at $x = 4$ or concavity changes

(b) $x = 2$

EITHER g' goes from negative to positive

OR $g'(2) = 0$ **and** $g''(2)$ is positive

(c)



Note: *a cubic curve through (4,0), M at x=2, P at (4,0).*

OPTIMIZATION

16. Let x be one side of the rectangle.

The other side will be $y = \frac{4a - 2x}{2} = 2a - x$

Then the area is given by $A = x(2a - x) = 2ax - x^2$

$\frac{dA}{dx} = 2a - 2x = 0 \Leftrightarrow x = a$

$\frac{d^2A}{dx^2} = -2 < 0$ so $x = a$ gives a maximum.

It is a square of side $x = a$ and the maximum area is $A = a^2$

17. Let x be one side of the rectangle.

The other side will be $y = \frac{a^2}{x}$.

Then the perimeter is given by $P = 2x + \frac{2a^2}{x}$

$\frac{dP}{dx} = 2 - \frac{2a^2}{x^2} = 0 \Leftrightarrow x^2 = a^2 \Leftrightarrow x = a$

$\frac{d^2P}{dx^2} = \frac{400}{x^3} > 0$ for $x = a$, so it gives a minimum.

It is a square of side $x = a$ and the minimum perimeter is $P = 4a$

18. (a) $6x^2+6y^2=300 \Leftrightarrow x^2+y^2=50 \Leftrightarrow y=\sqrt{50-x^2}$

(b) $V=x^3+(50-x^2)^{\frac{3}{2}}$.

$$\frac{dV}{dx}=3x^2-\frac{3}{2}(50-x^2)^{1/2}2x=3x^2-3x\sqrt{50-x^2}$$

$$\frac{dV}{dx}=0 \Leftrightarrow 3x^2-3x\sqrt{50-x^2}=0 \Leftrightarrow x=\sqrt{50-x^2} \Leftrightarrow x^2=50-x^2 \Leftrightarrow x=5$$

Then $y=5$. We have two similar cubes of total volume $V=5^3+5^3=250$.

Notice: By using GDC, graph mode, the minimum of the function $V=x^3+(50-x^2)^{\frac{3}{2}}$ is (5,250)

So the minimum value is 250

19. METHOD 1

(a) Let $D=\sqrt{(a-2)^2+(a^2-\frac{1}{2})^2}$

(b) $\frac{dD}{da}=\frac{1}{2\sqrt{---}}\left[2(a-2)+2(a^2-\frac{1}{2})2a\right]=\frac{1}{2\sqrt{---}}\left[2a-4+4x^3-2a\right]=\frac{1}{2\sqrt{---}}\left[4a^3-4\right]$

$=\frac{2}{\sqrt{---}}\left[a^3-1\right]$

(c) $\frac{dD}{da}=0 \Leftrightarrow a^3-1=0 \Leftrightarrow a=1$

(i) The point is $(1, 1^2)$ i.e. (1, 1)

(ii) The minimum distance is $D=\frac{\sqrt{5}}{2}$ $(\cong 1.12)$

Notice :

we can also use the GDC graph for the function $D=\sqrt{(x-2)^2+(x^2-\frac{1}{2})^2}$

It has a minimum at (1, 1.12)

Hence (i) The point is $(1, 1^2)$ i.e. (1, 1)

(ii) The minimum distance is D = 1.12

20. (a) $x^2y=125 \Leftrightarrow y=\frac{125}{x^2}$

(b) $S=2x^2+4xy=2x^2+4x\frac{125}{x^2}=2x^2+\frac{500}{x}$

(c) $\frac{dS}{dx}=4x-\frac{500}{x^2}$

$$4x-\frac{500}{x^2}=0 \Leftrightarrow 4x=\frac{500}{x^2} \Leftrightarrow x^3=125 \Leftrightarrow x=5$$

$$\frac{d^2S}{dx^2}=4+\frac{1000}{x^3}$$

For $x=5$, $\frac{d^2S}{dx^2}<0$ hence max

$S_{\max}=150$

B. Paper 2 questions (LONG)

21. (a) $f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

(b) $f(1) = 4 \Rightarrow a + b + c = 4$

$f'(1) = 0 \Rightarrow 3a + 2b + c = 0$

$f''(2) = 0 \Rightarrow 12a + 2b = 0$

(c) $a = 1, b = -6, c = 9$

(d) $f'(x) = 3x^2 - 12x + 9$, stationary points: $x = 1$, $x = 3$

minimum at $x = 3$ since $f''(x) = 6x - 12$ and $f''(3) = 6 > 0$

22. (a) (i) $f'(x) = \dfrac{\left(x \times \dfrac{1}{2x} \times 2\right) - (\ln 2x \times 1)}{x^2} = \dfrac{1 - \ln 2x}{x^2}$

(ii) $f'(x) = 0 \Leftrightarrow \dfrac{1 - \ln 2x}{x^2} = 0$ only at 1 point, when $x = \dfrac{e}{2}$

(iii) Maximum point when $f'(x) = 0$.

$f'(x) = 0$ for $x = \dfrac{e}{2}$ (= 1.36)

$y = f\left(\dfrac{e}{2}\right) = \dfrac{2}{e}$ (= 0.736)

(b) $f''(x) = \dfrac{-\dfrac{1}{2x} \times 2 \times x^2 - (1 - \ln 2x)2x}{x^4} = \dfrac{2\ln 2x - 3}{x^3}$

Inflexion point $\Rightarrow f''(x) = 0 \Rightarrow 2\ln 2x = 3 \Rightarrow x = \dfrac{e^{1.5}}{2}$ (= 2.24)

23. (a) B, D

(b) (i) $f'(x) = -2xe^{-x^2}$

(ii) product rule

$f''(x) = -2e^{-x^2} - 2x \times -2xe^{-x^2} = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)\,e^{-x^2}$

(c) $f''(x) = 0 \Leftrightarrow (4x^2 - 2) = 0$

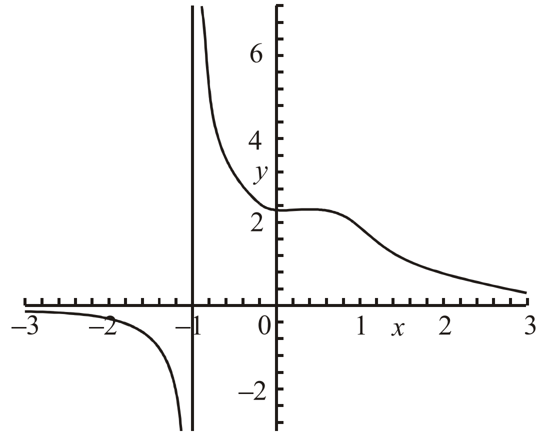
$p = 0.707\left(= \dfrac{1}{\sqrt{2}}\right)$, $q = -0.707 \left(= -\dfrac{1}{\sqrt{2}}\right)$

(d) checking sign of f'' on either side of POI

sign change of $f''(x)$

24. (a) (i) Vertical asymptote $x = -1$ (ii) Horizontal asymptote $y = 0$

(iii)



(b) (i) $f'(x) = \dfrac{-6x^2}{(1+x^3)^2}$

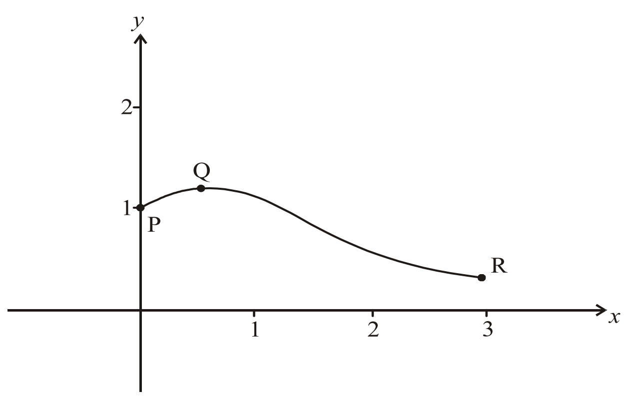
$$f''(x) = \frac{(1+x^3)^2(-12x) + 6x^2(2)(1+x^3)^1(3x^2)}{(1+x^3)^4}$$

$$= \frac{(1+x^3)(-12x) + 36x^4}{(1+x^3)^3} = \frac{-12 - 12x^4 + 36x^4}{(1+x^3)^3} = \frac{12x(2x^3 - 1)}{(1+x^3)^3}$$

(ii) Point of inflexion => $f''(x) = 0$ => $x = 0$ or $x = \sqrt[3]{\frac{1}{2}}$

$x = 0$ or $x = 0.794$ (3 sf)

25. (a)



(b) (i) $f'(x) = 2e^{-x} + (2x + 1)(-e^{-x}) = (1 - 2x)e^{-x}$

(ii) At **Q**, $f'(x) = 0$

$x = 0.5$, $y = 2e^{-0.5}$ **Q** is $(0.5, 2e^{-0.5})$

(c) $1 \le k < 2e^{-0.5}$

(d) $f''(x) = 0 \Leftrightarrow e^{-x}(-3 + 2x) = 0$

This equation has only one root. So f has only one point of inflexion.

26. (a) $x = 1$

(b) (i) $f(-1000) = 2.01$ (ii) $y = 2$

(c) $f'(x) = \dfrac{(x-1)^2(4x-13) - 2(x-1)(2x^2 - 13x + 20)}{(x-1)^4}$

$$= \frac{(4x^2 - 17x + 13) - (4x^2 - 26x + 40)}{(x-1)^3} = \frac{9x - 27}{(x-1)^3}$$

(d) $f'(3) = 0$ $\Rightarrow$ stationary point

$f''(3) = \dfrac{18}{16} > 0$ $\Rightarrow$ minimum

(e) Point of inflexion $\Rightarrow f''(x) = 0 \Rightarrow x = 4$

$x = 4 \Rightarrow y = 0$ $\Rightarrow$ Point of inflexion = (4, 0)

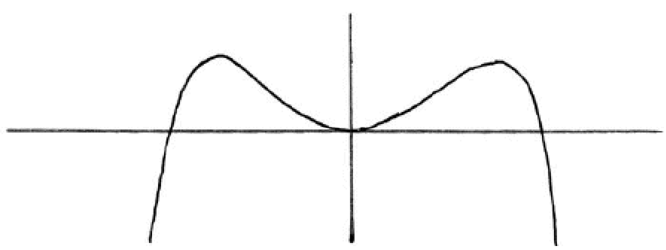
27. (a) (i) –1.15, 1.15

(ii) it occurs at P and Q (when $x = -1.15$, $x = 1.15$)

$k = -1.13$, $k = 1.13$

(b) $g'(x) = x^3 \times \dfrac{-2x}{4-x^2} + \ln(4-x^2) \times 3x^2 = \dfrac{-2x^4}{4-x^2} + 3x^2 \ln(4-x^2)$

(c)



$w = 2.69$, $w < 0$

28. (a) (i) coordinates of A are (0, –2)

(ii) $f(x) = 3 + 20 \times (x^2 - 4)^{-1}$

$f'(x) = 20 \times (-1) \times (x^2 - 4)^{-2} \times (2x) = -40x(x^2 - 4)^{-2}$

OR $\dfrac{(x^2 - 4)(0) - (20)(2x)}{(x^2 - 4)^2}$

substituting $x = 0$ into $f'(x)$ gives $f'(x) = 0$

(b) (i) $f'(0) = 0$ (stationary)

$f''(0) = \dfrac{40 \times 4}{(-4)^3}\left(= \dfrac{-5}{2}\right)$ negative

then the graph must have a local maximum

(ii) $f''(x) = 0$ at point of inflexion,

but the second derivative is never 0 (the numerator is always positive)

(c) getting closer to the line $y = 3$, horizontal asymptote at $y = 3$

(d) $y \le -2$, $y > 3$

29. (a) $f'(x) = e^x(1 - x^2) + e^x(-2x) = e^x(1 - 2x - x^2)$

(b) $y = 0$

(c) at the local maximum or minimum point

$f'(x) = 0 \Leftrightarrow e^x(1 - 2x - x^2) = 0 \Rightarrow 1 - 2x - x^2 = 0$

$r = -2.41$ $s = 0.414$ (**OR** directly by GDC graph)

(d) $f'(0) = 1 \Rightarrow$ gradient of the normal $= -1$

$y - 1 = -1(x - 0) \Leftrightarrow x + y = 1$

(e) (i) intersection points at (0,1) and (1,0)

30. (a) $f'(x) = x^2 - 2x - 3$

$x^2 - 2x - 3 = 0 \Leftrightarrow x = \dfrac{2 \pm \sqrt{16}}{2} \Leftrightarrow x = -1$ or $x = 3$

$x = -1$ (ignore $x = 3$) $y = -\dfrac{1}{3} - 1 + 3 = \dfrac{5}{3}$

coordinates are $\left(-1, \dfrac{5}{3}\right)$

(b) (i) (–3, –9)

(ii) (1, –4)

(iii) reflection gives (3, 9) stretch gives $\left(\dfrac{3}{2}, 9\right)$

31. (a) quotient rule

$f'(x) = \dfrac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \dfrac{-1}{\sin^2 x}$

(b) $f'(x) = -(\sin x)^{-2}$

$f''(x) = 2(\sin x)^{-3}(\cos x)$ $\left(= \dfrac{2\cos x}{\sin^3 x}\right)$

(c) substituting $\dfrac{\pi}{2} \Rightarrow p = -1,\ q = 0$

(d) second derivative is zero, second derivative changes sign

32. (a) $\dfrac{dy}{dx} = e^x(\cos x + \sin x) + e^x(-\sin x + \cos x) = 2e^x \cos x$

(b) $\dfrac{dy}{dx} = 0 \Rightarrow 2e^x \cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \dfrac{\pi}{2} \Rightarrow a = \dfrac{\pi}{2}$

$y = e^{\frac{\pi}{2}}(\cos\dfrac{\pi}{2} + \sin\dfrac{\pi}{2}) = e^{\frac{\pi}{2}} \Rightarrow b = e^{\frac{\pi}{2}}$

(c) At D, $\dfrac{d^2y}{dx^2} = 0 \Rightarrow 2e^x \cos x - 2e^x \sin x = 0 \Rightarrow 2e^x(\cos x - \sin x) = 0$

$\Rightarrow \cos x - \sin x = 0 \Rightarrow x = \dfrac{\pi}{4}$

$y = e^{\frac{\pi}{4}}(\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{4}) = \sqrt{2}\, e^{\frac{\pi}{4}}$

33. (a) $y = 0$

(b) $f'(x) = \dfrac{-2x}{(1+x^2)^2}$

(c) $f'(x) = -2x(1+x^2)^{-2}$,

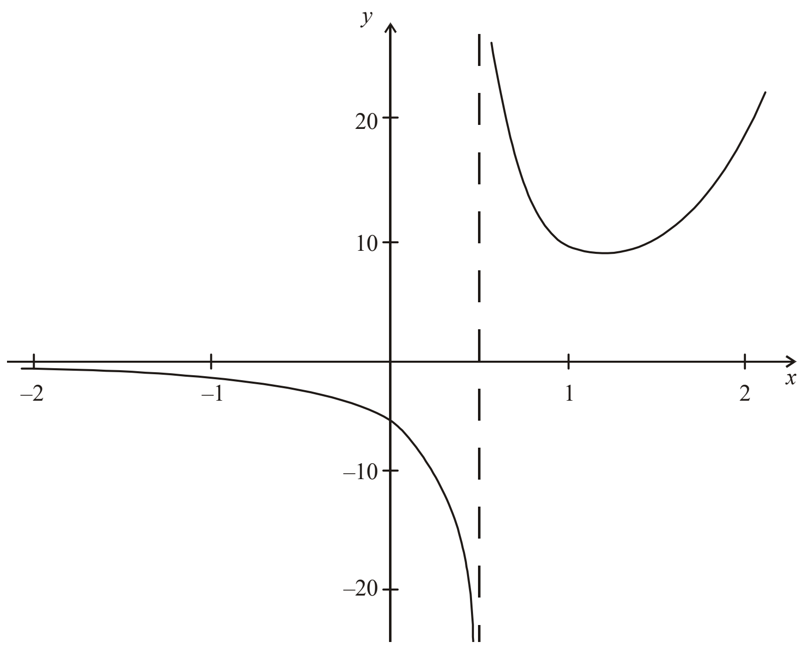
$$f''(x) = -2(1+x^2)^{-2} + 4x(1+x^2)^{-3}2x = \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}$$

$$= \frac{-2(1+x^2)}{(1+x^2)^3} + \frac{8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3}$$

(d) $f''(x) = 0 \Leftrightarrow \; 6x^2 - 2 = 0 \Leftrightarrow \; x = \pm\sqrt{\dfrac{1}{3}}$

The maximum gradient is at $x = \dfrac{-1}{\sqrt{3}}$

34. (a)



***Note:** left branch asymptotic to the x-axis, vertical asymptote at x* = 1/2

(b) $x = \dfrac{1}{2}$ (must be an equation)

(c) $f'(x) = 2e^{2x-1} - 10(2x-1)^{-2}$

(e) (i) $x = 1.11$ (accept (1.11, 7.49)) (ii) $p = 0$, $q = 7.49$ $(0 \le k < 7.49)$

35. (a) π

(b) (i) $f'(x) = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$

(ii) At B, $f'(x) = 0$

(c) $f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x = 2e^x \cos x$

(d) (i) At A, $f''(x) = 0$

(ii) $2e^x \cos x = 0 \Leftrightarrow \cos x = 0$

$x = \dfrac{\pi}{2}$, $y = e^{\frac{\pi}{2}}$ Coordinates are $\left(\dfrac{\pi}{2}, e^{\frac{\pi}{2}}\right)$